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THREE-BODY MEAN MOTION RESONANCES AND THE CHAOTIC STRUCTURE OF THE ASTEROID BELT

D. NESVORNÝ AND A. MORBIDELLI
Observatoire de la Côte d'Azur–Nice, B.P. 4229, Le Mont-Gros, F-06304 Nice Cedex 4, France
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ABSTRACT

We discuss the existence and the properties of a new type of mean motion resonance populated by a large number of asteroids, the resonant angle being defined as a linear combination (with integer coefficients) of the mean longitudes of asteroid, Jupiter, and *Saturn.* We call these resonances the *three-body mean motion resonances.* In the present paper, we show that the anomalous large oscillation of the mean semimajor axis of about 250 numbered asteroids is associated with some of the most prominent three-body mean motion resonances, and we conjecture that the actual number of the resonant asteroids is much larger. The positive Lyapunov exponent detected for the majority of resonant bodies in our numerical integrations suggests the chaotic nature of their orbits. Moreover, we show, using frequency analysis, that orbits in the three-body mean motion resonances may slowly and chaotically diffuse in eccentricity. The existence of such diffusion may have major consequences for our present understanding of the long-term evolution of the asteroidal belt and the delivery of objects to near-Earth orbits.

Key words: celestial mechanics, stellar dynamics — minor planets, asteroids

1. INTRODUCTION

In the last decade it has been shown that many asteroids in the main belt have a strongly chaotic motion with a very short Lyapunov time[1] of 10^3–10^5 yr (Milani & Nobili 1992; Milani 1993; Holman & Murray 1996; Milani, Nobili, & Knežević 1997; Šidlichovský & Nesvorný 1998). Several of these asteroids are evidently related to mean motion resonances of moderate order.[2] While mean motion resonances of low order (say, $2 \leq q \leq 4$) force the resonant asteroids to become planet crossers on a short timescale (of order of a few million years), and therefore are basically depleted of asteroids, in mean motion resonances of moderate order (say, $5 \leq q \leq 7$–9) the times required to become a planet crosser are much longer (from tens of millions of years to of order 1 Gyr), so that the resonances may be largely populated by asteroids of primordial and/or collisional origin. This has been shown by Murray & Holman (1997) for the resonances in the outer asteroid belt, but their result can be extended to the mean motion resonances of equivalent order in the main belt, i.e., for semimajor axis *a* ranging between 2.1 and 3.25 AU. Hence, the moderate-order resonances located in the main belt (such as 7:2, 10:3, 11:4, 8:3, 12:5, 9:4, 11:5, 13:6) are the places where most of chaotic asteroids should be expected.

However, there are also many asteroids with a short Lyapunov time of 5×10^3 to 10^5 yr, which may be difficultly related to any mean motion resonance of significant order. The main example is provided by the asteroid 490 Veritas (Milani & Farinella 1994). A tentative explanation of its behavior given by Milani et al. (1997) accounted for the effect of the 21:10 and 44:21 commensurabilities and periodic variations of the instantaneous perihelion velocity caused by the secular oscillation of asteroid's eccentricity. However, these resonances are of very high order, and as we will show in the following, they are not likely to produce the mean semimajor-axis oscillations of about 0.005 AU and the degree of chaoticity observed in the numerical integrations; moreover, they are not exactly in the region spanned by the semimajor-axis oscillations, and the latter are not correlated with librations of the critical angle(s).

We present here an alternative explanation of 490 Veritas's behavior based on the consideration of a resonance belonging to a class that, up to now, has been a priori neglected in asteroid science (with the only exception being a short paper by Fernández & Beaugé 1988): that of the *three-body mean motion resonances.* By this term, we mean the resonances that involve not only the mean motions of asteroid and Jupiter but also the mean motion of Saturn, corresponding to the relation

$$m_J \dot{\lambda}_J + m_S \dot{\lambda}_S + m\dot{\lambda} \sim 0 \,, \quad (1)$$

where $\dot{\lambda}_J$, $\dot{\lambda}_S$, and $\dot{\lambda}$ denote the mean motions of Jupiter, Saturn, and the asteroid, respectively, and m_J, m_S, and m are integers. The resonant angles associated with such a resonance turn out to be any combination

$$\sigma_{p_J, p_S, p} = m_J \lambda_J + m_S \lambda_S + m\lambda + p_J \varpi_J + p_S \varpi_S + p\varpi \,, \quad (2)$$

where λ_J, λ_S, λ are the mean longitudes and ϖ_J, ϖ_S, ϖ are the perihelion longitudes of Jupiter, Saturn, and asteroid, and p_J, p_S, and p are integers that fulfill the d'Alembert rule, $m_J + m_S + m + p_J + p_S + p = 0$.

Since the perihelion longitudes ϖ_J, ϖ_S, and ϖ have a small but nonzero frequency, angles $\sigma_{p_J, p_S, p}$ with different p_J, p_S, and p have zero derivative at different locations. Therefore, a three-body mean motion resonance of type $m_J \dot{\lambda}_J + m_S \dot{\lambda}_S + m\dot{\lambda} \sim 0$ splits in a natural way into a "multiplet" of resonances—the separation between them being only a few times 10^{-4} AU—each corresponding to a different perihelia

[1] The Lyapunov time is the inverse of the maximum Lyapunov exponent. It is the mean time span on which two initially close chaotic orbits increase their mutual distance by the value of the base of the natural logarithm, $\sim$2.718.

[2] The mean motion resonances are usually denoted by the ratio of Jupiter and asteroid orbital periods, which is written as $(p + q){:}p$, with p, q integers; q is called the order of the resonance and determines its strength. In fact, the coefficient of the resonant terms turns out to be proportional to e^q, where e denotes the asteroid's eccentricity.

combination. In our terminology, by the name *three-body mean motion resonance* we will refer hereafter to the whole multiplet,[3] and a particular three-body mean motion resonance will be denoted by the integers $m_J\, m_S\, m$.

During the revisions of the manuscript, we have learned about the paper by Murray, Holman, & Potter (1998), independently submitted to this journal at about the same time as ours. They point out the importance of three-body resonances for the outer asteroid belt and analytically estimate the Lyapunov and diffusion times for the 6 −2 −3 and 3 1 −2 resonances. We suggest their article for a complementary reading on the subject—the reader may find in their text a numerical study of the effect of Saturn's perturbation and an analytic approach to modeling the three-body resonances' dynamics. Our slightly different analytic approach to the study of three-body resonance is described in Nesvorný & Morbidelli (1998).

We will start by presenting in the next section the numerical evidence that 490 Veritas and several other main-belt asteroids of similar behavior are actually in a three-body mean motion resonance. A numerical survey of the location and size of the main three-body resonances is presented in § 3 and reveals how our view of the dynamical structure of the asteroid belt is strongly modified by the consideration of the new class of resonances. Section 4 is devoted to listing the numbered asteroids, which we succeeded in relating to some three-body mean motion resonance by looking at the evolution of their critical angles; we will show that basically every numerically detectable chaotic region can be associated with either a usual mean motion and secular resonance or with a three-body resonance. Our conclusions are given in § 5.

2. WHY IS 490 VERITAS (AS WELL AS OTHERS) CHAOTIC?

We have numerically integrated the evolution of the orbit of 490 Veritas subjected to the perturbations of the four outer planets, using the symmetric multistep method (Quinlan & Tremaine 1990). The output has been sampled at 2 yr intervals.

Having conjectured, on the basis of its orbital frequency, that 490 Veritas is related to the three-body mean motion resonance 5 −2 −2, we have computed the evolution of the secular angle

$$\sigma = 5\lambda_J - 2\lambda_S - 2\lambda - \varpi \, . \qquad (3)$$

To average out short periodic oscillations, σ has been first transformed to $z = \exp i\sigma$ [$i = (-1)^{1/2}$], so that the application of a digital filter does not produce artificial results when the angle passes through the limits of its definition interval. Then the digital low-pass filter B of Nesvorný & Ferraz-Mello (1997; see Quinn, Tremaine, & Duncan 1991) was applied on z and on the semimajor axis. The filter was subsequently applied three times by increasing the sampling by a factor of 3 at every step, so that smoothed variables sampled at 54 yr intervals were obtained. By the effect of filtration, all the frequencies with periods less than approximately 100 yr have been canceled. The filtering is necessary in the Veritas case since the short periodic oscillations of the semimajor axis (of about 0.03 AU), forced by the vicinity of the 2:1 resonance, would hide its mean evolution on the given time interval. The libration of the resonant angle would be visible even without filtering, but the smoothed variable presents a better view.

Figure 1 shows the smoothed semimajor axis and resonant angle (eq. [3]) of 490 Veritas on the interval 2×10^5 yr. The observed evolution of the semimajor axis is very similar to that shown in Figure 3 of Milani et al. (1997). As was stated in that paper, the oscillation of $\sim 5 \times 10^{-3}$ AU cannot be explained on the basis of the standard secular theory (Milani, Nobili, & Carpino 1987). Milani at al. (1997) attempted to relate these oscillations to the 21:10 and 44:21 mean motion resonances with Jupiter and to the variation of the instantaneous perihelion frequency, which, according to the authors, would force Veritas to cross the resonances. Their reasoning was based on the apparent correlation between the semimajor-axis and eccentricity variations. In short, the secular oscillations of the asteroid's eccentricity force the instantaneous perihelion frequency to change, and this in turn forces the above resonances' locations to oscillate with respect to the asteroid position, and the repetitive passages through the resonances would presumably cause Veritas's semimajor-axis variations. In this argument, the correlation between the oscillations of semimajor axis and eccentricity is a necessary condition, and in the case of 490 Veritas, this condition is approximately fulfilled.

However, several counterarguments can be given here: (1) The mentioned resonances are of a very high order, and our analytic computation (based on the evaluation of the leading coefficients in the perturbing function) shows that the size of the resonant zone should be less than 5×10^{-5}

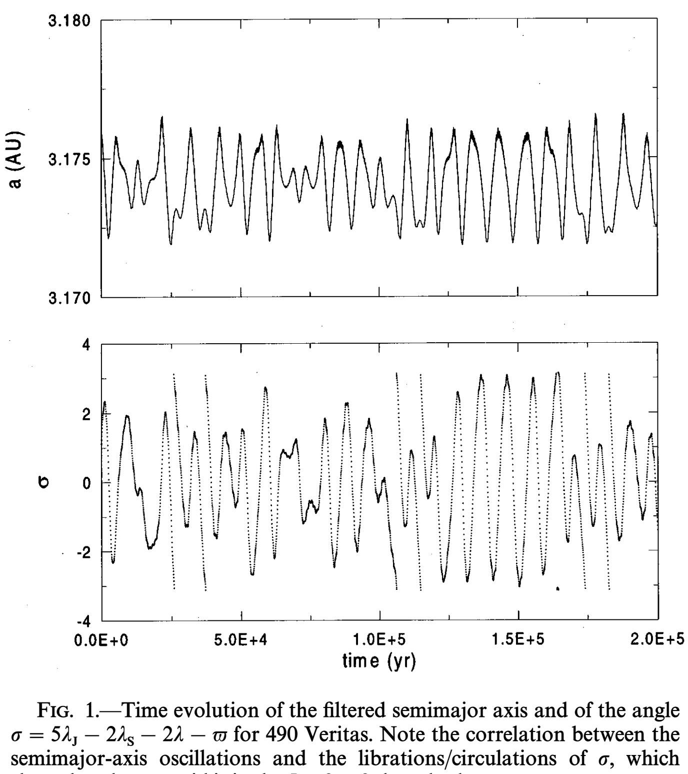

FIG. 1.—Time evolution of the filtered semimajor axis and of the angle $\sigma = 5\lambda_J - 2\lambda_S - 2\lambda - \varpi$ for 490 Veritas. Note the correlation between the semimajor-axis oscillations and the librations/circulations of σ, which show that the asteroid is in the 5 −2 −2 three-body resonance.

[3] As a matter of fact, the longitudes of node should also be considered in eq. (2) in the case where the inclinations are not neglected, thus yielding a much larger number of multiplet terms. However, in our exploration these new terms do not seem to be of much importance, so that, in the first approximation, we will concentrate on the planar case and consider the resonant angles defined with only the longitudes of perihelia as denoted by eq. (2).

AU for the 21:10 resonance.[4] The 44:21 resonance is even thinner, by several orders of magnitude. Consequently, the effect of both resonances must be very, very small. (2) The location of the 21:10 resonance is not exactly centered in the region spanned by Veritas's semimajor-axis oscillations; in fact, the 12 resonances forming the multiplet associated with the 21:10 resonance are located in the interval 3.1708–3.1724 AU, while Veritas oscillates between 3.1719 and 3.1765 AU. (3) The oscillations of Veritas's semimajor axis are not correlated with the circulation/libration of any of the critical angles associated with the 21:10 and 44:21 resonances. (4) The correlation between semimajor axis and eccentricity is only approximately verified for 490 Veritas; Veritas is nevertheless a particularly "good" case in this respect, because it has a relatively high perihelion frequency, $\sim 138'' \ \mathrm{yr}^{-1}$. In most of the other cases of "stable chaos" listed by Milani et al. (1997) in other regions of the asteroid belt, this condition is not fulfilled at all. Thus, even if the Milani et al.'s explanation could be accepted for 490 Veritas, it would necessarily fail in other cases. (5) A quantitative model based on the Milani et al. scenario has failed to explain Veritas's behavior (Lemaître 1997).

We claim that the 5 −2 −2 three-body mean motion resonance dominates the local dynamics and is a much more convincing explanation of Veritas's behavior. Figure 1 shows clearly that the angle (eq. [3]) is a very slow angle with several librations of about 10^4 yr around zero, alternating with periods of slow circulation. Moreover, its evolution is very well correlated with the evolution of the semimajor axis. This shows that 490 Veritas is in fact in the three-body mean motion resonance, with the resonant angle defined by equation (3). It should also be noted that the irregular behavior of both the semimajor axis and the resonant angle witness the fact that the motion of 490 Veritas is chaotic. Indeed, its Lyapunov time estimated by Milani et al. is about 8500 yr.

We will now show that the three-body mean motion resonances also provide the explanation for most of the chaotic asteroids in the main belt listed in Table 1 of Milani et al. (1997). Figure 2 shows the result concerning 485 Genua and the resonance 3 −1 −1. Once again, the resonant angle $\sigma = 3\lambda_J - \lambda_S - \lambda - \varpi$ is very slowly variable. It librates around π up to approximately 8×10^4 yr and switches to the prograde circulation at 10^5 yr. From then on, it is in the proximity of the 13:5 resonance, which is located at 2.7510 AU. But the 3 −1 −1 three-body resonance still governs the dynamical behavior in this region. This is clear from the fact that the oscillations of the semimajor axis are very well correlated with the behavior of the resonant angle, not only up to 8×10^4 yr (when the amplitude of oscillation is about 3×10^{-3} AU[5] and σ librates) but also later, when σ circulates. In integrations covering a longer time span, the alternation between libration and oscillation of the resonant angle σ happens often, which is clear evidence that the orbit is chaotic (the Lyapunov time is even shorter than that of 490 Veritas: it is about 6500 yr).

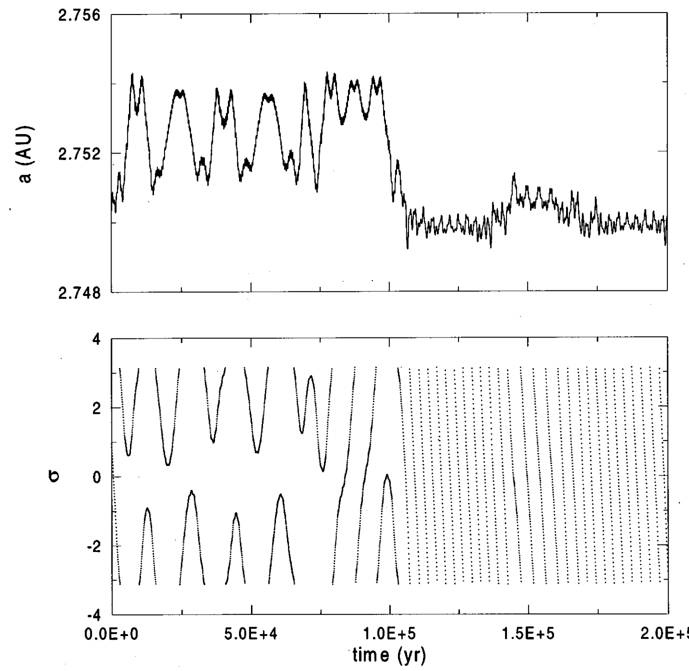

FIG. 2.—Same as Fig. 1, but for the asteroid 485 Genua. In this case, $\sigma = 3\lambda_J - \lambda_S - \lambda - \varpi$. The asteroid is initially inside the 3 −1 −1 three-body resonance.

Similarly, we have succeeded in associating 10 Hygiea with the 8 −4 −3 three-body resonance; 221 Eos with the 6 2 −3 (but here the secular resonance $g + s - g_6 - s_6 = 0$ is also important, where g and s are the mean asteroid perihelion and node frequencies, respectively, and g_6 and s_6 are two of the secular planetary frequencies; Milani & Knežević 1992); and 564 Dudu and 795 Fini with the 3 −1 −1.

Thus, we can conclude that the concept of the three-body mean motion resonances is the one that was missing in order to explain the observed chaotic behavior of many real asteroids (we will list about 250 of them in § 4, just among the numbered asteroids), which has been considered so puzzling in several recently published papers. From now on, the three-body mean motion resonance should be considered, together with the ordinary mean motion and secular resonances, among the main chaos generators in the asteroidal belt.

3. A NUMERICAL SURVEY OF THE MAIN THREE-BODY RESONANCES

The three-body mean motion resonances are more dense in phase space than the ordinary mean motion resonances with Jupiter, because three mean motion frequencies instead of two are involved. This is shown in Figure 3, where we mark with vertical lines the locations of both the ordinary and the three-body mean motion resonances, for semimajor axis ranging all over the asteroid belt (2–4 AU). For the ordinary mean motion resonances (*heavy lines*), the height of the bars is given as $10 - q$, where q is the order in eccentricity of the coefficient of the leading resonant harmonic, so that all the resonances up to order 9 are reported. For the three-body mean motion resonances (*light lines*), the height of the bars is instead given as $7 - q$. This is done to account for the fact that the coefficients of the three-body resonant harmonics are quadratic in the masses (see Nesvorný & Morbidelli 1998) and that for an eccentricity

[4] The resonant size is given for the proper eccentricity of 490 Veritas, which is about 0.0652. It is in fact the size of the strongest multiplet component $21\lambda_J - 10\lambda - 7\varpi_J - 4\varpi$ at this eccentricity, the component with the maximum number of asteroid perihelia (and also with other perihelia combinations), i.e., $21\lambda_J - 10\lambda - 11\varpi$, being much weaker, of size $\sim 2.5 \times 10^{-6}$ AU.

[5] In this case, the short periodic oscillations of the semimajor axis canceled by the filtering procedure were about 8×10^{-3} AU.

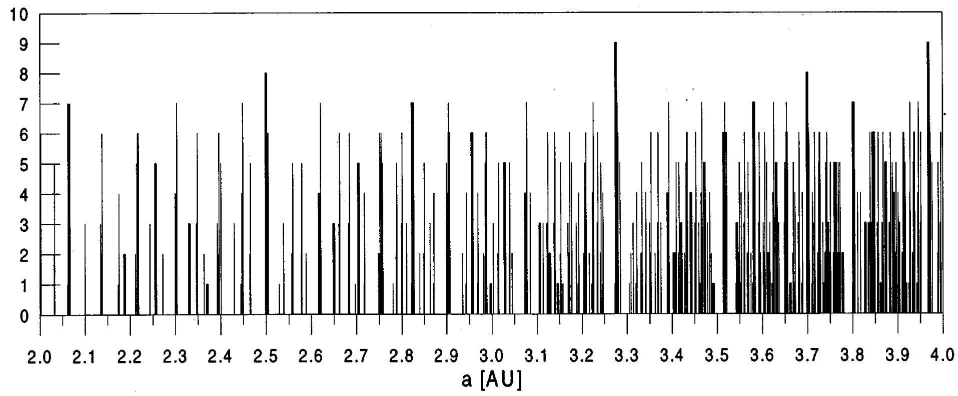

FIG. 3.—Location of the ordinary mean motion resonances (*heavy lines*) and the three-body resonances (*light lines*). The height of each bar is related to the resonance order, as explained in the text, so that the vertical scale gives a comparative indication of the strength of the resonances.

$e = 0.066$ the mass of Saturn is equal to e^3; in other words, a three-body resonance of a given order q should have roughly the same strength as a usual resonance of order $q + 3$ for eccentricity of about 0.05–0.10. With this trick, the vertical scale of Figure 3 gives a comparative indication of the strength of the various resonances, so that the reader can appreciate at a glance the surprising overdensity of the three-body resonances. Thus, the three-body mean motion resonances seem to be the main actors structuring the dynamics in the main asteroid belt. A rough estimate shows that the resonances at level 1 in the scale discussed above have an expected width in semimajor axis of order 10^{-4} AU at $e \sim 0.1$.

Figure 3 provides only a qualitative indication of the resonances' strength, since it does not account for the distance from the main perturber (the resonances of a given order that are far from Jupiter are expected to be narrower than the resonances of the same order but close to Jupiter). A more reliable numerical measure of the width of 17 three-body resonances is reported in Table 1, which is based on the numerical integrations of the evolution of a number of resonant asteroids. As for 490 Veritas and 485 Genua (Figs. 1 and 2), the asteroids have been integrated for 2×10^5 yr, taking into account the perturbations of four outer planets and filtering the oscillations with periods less than ~ 100 yr, so that the usually very large short-period variations of the semimajor axis have been completely eliminated.

For each considered three-body resonance, Table 1 lists by the catalog number one or more resonant asteroids that we have numerically integrated. The following columns report the proper eccentricity (e_p) of the considered asteroids, the center value (a_c) and the amplitude (δa) of their semimajor-axis oscillation, the typical period of these oscillations ($T_{\rm var}$), and the libration center of the resonant angle (σ_c).

Several qualifications are in order: (1) The amplitude of the semimajor-axis oscillations is an indication of the width of the resonance only for a value of the eccentricity close to the asteroid's e_p. Note that, the motion of most asteroids being chaotic, such an amplitude may not be well determined on the basis of a 2×10^5 yr integration. Moreover, even if some quasi-regular librations of the resonant angle occur, the width of the resonance may be larger than the observed semimajor-axis oscillations; in fact, in an integrable case, the width of the resonance is equal to the amplitude of semimajor-axis oscillation only if the libration

TABLE 1

NUMERICAL RESULTS ON THE THREE-BODY RESONANCES

Resonance	Asteroid	e_p	a_c (AU)	δa (10^{-3} AU)	$T_{\rm var}$ (10^3 yr)	σ_c
4 −1 −1......	2440	0.1113	2.2157	0.6	50	0
4 −2 −1......	463	0.1795	2.3977	3.0	~10	0
7 −2 −2......	1966	0.1241	2.4476	0.6	~30	...
7 −3 −2......	1430	0.1741	2.5599	0.5	~30	π
2 2 −1......	258	0.1687	2.6155	0.7	~20	...
6 −1 −2......	53	0.2092	2.6190	~1.0	35	π
	3426	0.0961	2.6195	~1.0	~40	...
4 −3 −1......	792	0.1604	2.6230	2.5	25	...
7 −4 −2......	789	0.1471	2.6857	0.5	20	...
3 −1 −1......	485	0.1958	2.7525	3.0	15	π
	1642	0.0964	2.7525	2.0	25	π
4 −4 −1......	22	0.0881	2.9095	~1.0	~50	...
5 −1 −2......	576	0.1758	2.9860	2.0	20	0
3 −2 −1......	2395	0.0690	3.0790	4.0	10	π
6 1 −3......	936	0.1540	3.1385	0.4	10	...
8 −4 −3......	10	0.1347	3.1418	0.5	~30	...
3 3 −2......	106	0.1466	3.1708	~2.0	...	...
5 −2 −2......	490	0.0652	3.1738	4.0	10	0
	2039	0.1567	3.1743	4.0	3.8	0
	3460	0.2031	3.1745	5.0	3.6	0
7 −2 −3......	530	0.1937	3.2080	<8.0	12	π

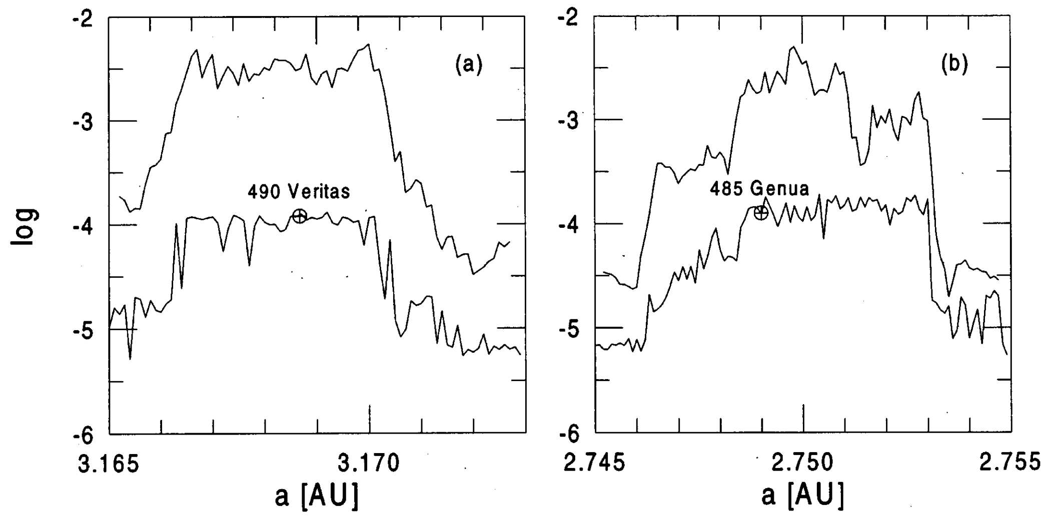

FIG. 4.—Maximum Lyapunov exponent (*lower lines*) and the relative change of the perihelion frequency (*upper lines*) as a function of the semimajor axis for the 5 −2 −2 and 3 −1 −1 three-body resonances. The positions of 490 Veritas and 485 Genua are also reported. See text for explanation.

amplitude of the resonant angle is equal to 180°. Nevertheless, we believe that the numbers reported in the δa column of Table 1 indicate the real width of the resonances, within a factor 1.5 (the values listed with " ∼ " being less accurate than the others). Only in the case of 530 Turandot is the reported value just an upper limit on the expected width of the 7 −2 −3 resonance, because the resonant semimajor-axis oscillations were hidden by the short-period variations, which have some components with larger period than the cutoff of our filtering procedure, due to the close proximity to the 2:1 resonance. (2) The period $T_{\rm var}$ corresponds to the libration period in the case of nice librations of the resonant angle, and to the approximate period of libration/circulation otherwise. It depends not only on the resonance, but also on the eccentricity and libration amplitude of the considered asteroid. (3) The center of libration σ_c of the resonant angle is reported only when some well-defined librations are visible in the numerical integrations; it is not reported in cases of strongly chaotic motion.

Table 1 shows that the considered three-body mean motion resonances have all widths ranging from some 10^{-4} to about 5×10^{-3} AU. The typical timescale of the variation is of order 10^4 yr, although some asteroids may present faster oscillations (for instance, 3460 Ashkova, in 5 −2 −2, has a period of about 3600 yr). Note that the period of libration/circulation is expected to be of the same order as the Lyapunov time. This can be understood on the basis of a simple argument by Benettin & Gallavotti (1986; also illustrated in Morbidelli & Froeschlé 1996). The libration center of the resonant angle is either 0 or π, but several exceptions were observed, with temporary librations around different values.

To have a deeper insight of the chaotic structure associated with some three-body resonances, we have performed numerical integrations of a large number of test particles, perturbed by the four outer planets, with initial conditions on grids in semimajor axis at constant eccentricity and inclination. For each test particle, we have estimated the maximum Lyapunov exponent (MLE) and the rate of change of the perihelion frequency. As before, the integrations were performed using the symmetric multistep method (Quinlan & Tremaine 1990). The step size was 5 days, and the length of integration ranged from 2 to 4 Myr. In parallel with the propagation of the evolution of test particles and planets, the integration of the variational equations allowed us to estimate the MLE, and the application of digital filtering and of sophisticated Fourier analysis (Šidlichovský & Nesvorný 1997) allowed for the determination of the proper frequencies and the estimation of their rates of change (Laskar 1990).[6]

Figure 4 shows the results obtained for two cases: Figure 4*a* concerns the 5 −2 −2 resonance in the vicinity of 490 Veritas, and Figure 4*b* refers to the 3 −1 −1 resonance for the initial conditions near 485 Genua. In both cases the initial angles, inclination, and the eccentricity of the test particles were chosen equal to those of Veritas and Genua, respectively. The lower line in both graphs shows the maximum Lyapunov exponent as a function of the initial semimajor axis, while the upper line shows the relative change of the perihelion frequency over 1 Myr (both given in logarithmic scale).

In Figure 4*a*, the value of the MLE attains a plateau at $\sim 10^{-4}$ yr^{-1} between 3.165 and 3.17 AU, which is related to the 5 −2 −2 three-body resonance. Note that the width of this area is about 0.005 AU, basically the same as the amplitude of the semimajor-axis oscillation of 490 Veritas (Fig. 1, Table 1). The relative change of perihelion frequency is well correlated with the MLE profile. Recall that a positive rate of change is a reliable indicator that chaotic diffusion is at work (Laskar 1990). In the case of 490 Veritas the relative change of the perihelion frequency is about $10^{-2.5}$ Myr^{-1}, and in fact Milani & Farinella (1994) first showed with long-term numerical integrations that the asteroid diffuses in both eccentricity and inclination, while keeping its semimajor axis always in the range reported above.

In Figure 4*b*, the profiles of MLE and frequency change are similar to those of Figure 4*a*, but the plateaus are not so well defined. The MLE attains its largest value ($\sim 10^{-4}$ yr^{-1}) over 2.749–2.753 AU. Once again, this width is very close to the amplitude of Genua's semimajor-axis oscillations (Fig. 2, Table 1) and is well correlated with higher

[6] The package of programs, written in the C language, is available on demand.

diffusion speed (of order $\sim 10^{-3}$ Myr^{-1}).

Similar experiments have been done for other three-body resonances. The Lyapunov times are usually in the range from 5×10^3 to 10^5 yr. The relative frequency change is always well correlated with the MLE profile and exceeds 10^{-4} Myr^{-1}. This evidences the fact that the three-body resonances are quite powerful sources of chaotic behavior, and that a nonnegligible diffusion of eccentricity and inclination should be associated with all of them.

An additional interesting question concerns the origin of the chaotic behavior in the three-body mean motion resonances found in our numerical integrations. We believe that it is the overlap of different multiplet components (different p_J, p_S, p in eq. [2]) that plays a crucial role here. The multiplet components are wider than their separation and strongly interact, giving rise to the chaotic motion. The resonant orbit is then free to wander over the range of components of the multiplet that are overlapped.

An analytic model of three-body mean motion resonances, showing the origin of chaos, is developed in Nesvorný & Morbidelli (1998). Analytic formulae for the estimates of the Lyapunov and diffusion times may be found in Murray et al. (1998).

4. SYSTEMATIC IDENTIFICATION OF THE ASTEROIDS IN THREE-BODY RESONANCES

In order to identify among the population of known asteroids those that are associated with ordinary or three-body mean motion resonances, we have proceeded as follows.

We have based our identifications on the catalog (kindly provided to us by C. Froeschlé & R. Gonczi) of the 836 numbered asteroids with a *fast Lyapunov indicator* (Lega & Froeschlé 1997) greater than 10^8. This selection has been done by Froeschlé, Gonczi, & Lega (1997) by integrating over 50,000 yr the first 5400 numbered asteroids under the perturbations of the four outer planets. Such a time span would be too short for the correct computation of the maximum Lyapunov exponent, but the evaluation of the fast Lyapunov indicator allows (with some uncertainty) the detection of the most chaotic orbits.

In Figure 5, the points show the position of the 836 asteroids with respect to semimajor axis, eccentricity, and inclination (we used the proper elements by Milani & Knežević 1994 when available, and the osculating elements otherwise). In the bottom panel we indicate the location of some of the main ordinary (*heavy lines*) and three-body (*light lines*) resonances. The height of the resonant lines has been scaled as in Figure 3; however, note that, with respect to Figure 3, only some of the resonances have been plotted, as otherwise Figure 5 would be unreadable. Above each resonant line we report the integer coefficients defining the resonant angles: m_S m_S m denote the three-body resonance with angle $m_J \lambda_J + m_S \lambda_S + m\lambda + \cdots$, while m_J/m denotes the ordinary resonance with angle $m\lambda - m_J \lambda_J + \cdots$, where the points stand for a correct combination of longitudes of perihelia satisfying the d'Alembert rule.

Some asteroidal concentrations visible in Figure 5 can be easily identified with ordinary and three-body resonances of moderate order. The ordinary mean motion resonances 7:2, 10:3, 11:4, 8:3, 9:4, 13:6, etc., and the three-body mean motion resonances 4 −1 −1, 3 1 −1, 4 −2 −1, 5 3 −2, 5 2

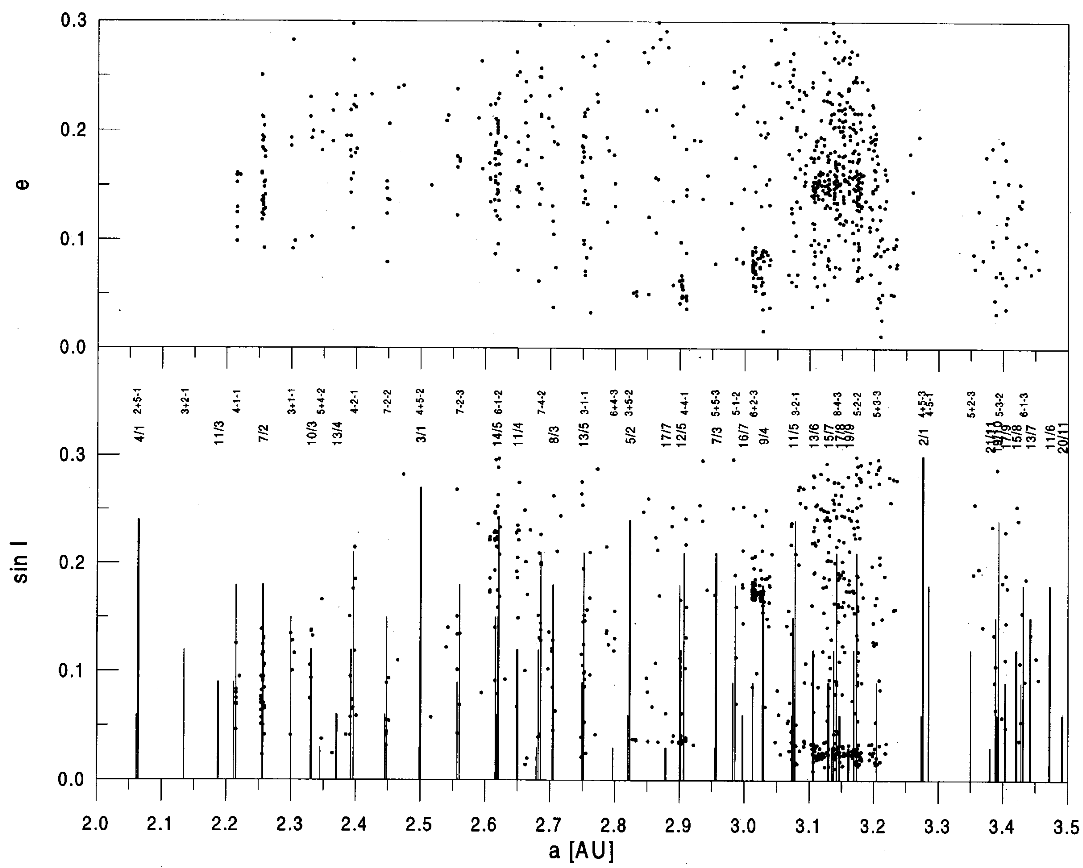

FIG. 5.—Chaotic asteroids with the fast Lyapunov indicator larger than 10^8 (*points*) and the location of the main ordinary and three-body mean motion resonances. The height of the resonant bars is scaled as in Fig. 3, but only some of the resonances are reported.

−2, 2 2 −1, 4 −3 −1, 5 1 −2, 3 −1 −1, 6 2 −3, 5 −2 −2, 5 3 −3, etc., seem to be responsible for most of the chaotic orbits of the real asteroids in the belt.

We stress that not all the important resonances are shown in Figure 5; for example, from the 28 points that seem to be associated with the 7:2 resonance, only 11 asteroids are actually related to this resonance and 13 asteroids are inside the very close (but not reported in Fig. 5) 9 −5 −2 and 5 5 −2 three-body resonances, as shown in Table 3. This shows that a more precise identification of the resonant asteroids is needed. To this purpose, we have integrated the chaotic asteroids over 10^5 yr and monitored the evolution of their filtered resonant angles (the integration was performed with four outer planets, and the results were checked in several cases integrating with seven planets). On the basis of these integrations, we have associated an asteroid with a resonance if (1) the corresponding resonant angle shows evident librations during the integration time span or (2) the resonant angle circulates with a period longer than several thousand years. In the latter case our experience shows that librations will start later, revealing the resonant dynamical nature of the body.

Table 2 lists the three-body mean motion resonances with the largest number of asteroids: the first and the second entries under each resonance label report the asteroids that satisfy, respectively, criteria 1 and 2 for resonant membership that we discussed above.

Table 3 lists the three-body resonances with fewer than 11 asteroids; in this case we have not distinguished between bodies satisfying the different criteria. Similarly, the asteroids in the ordinary mean motion resonances are listed in Tables 4 and 5.

We do not claim that the list of resonant asteroids identified by our method is in any regard complete. Even if we have probably succeeded in identifying most of the cases in the 836-asteroid file that can be associated with one of the studied 32 three-body and 10 ordinary mean motion resonances, there must be a large number of unidentified cases related to other mean motion resonances. Also, many possible candidates may have been lost in the reduction from 5400 to 836 asteroids based on the value of the fast Lyapunov indicator. Moreover, the initial 5400 asteroids constitute only one-sixth of all the cataloged asteroids at the present time. Thus, if our identification process resulted in finding 255 asteroids in the three-body resonances, the real number of the cataloged resonant asteroids should be (most pessimistically) about 1500 and very probably on the order of several thousand. Similarly, having identified 63 asteroids in the ordinary resonances, we estimate a lower limit of ∼380 resonant members, which is probably exceeded by several times because of the obvious incompleteness of our identification method. Therefore, the important conclusion of our search for resonant asteroids is that the expected populations, especially in the case of three-body resonances, are very numerous.

TABLE 2

NUMBERED ASTEROIDS IN THE THREE-BODY MEAN MOTION RESONANCES: RESONANCES WITH MORE THAN 10 IDENTIFIED MEMBERS

Resonance	Members
5 −2 −2 (45 members):	
Criterion 1......	490, 511, 744, 818, 1072, 1209, 1546, 1633, 1701, 1731, 1761, 2039, 2142, 2164, 2184, 2211, 2250, 2492,2515, 2587, 2615, 2657, 2666, 2670, 2731, 2863, 3204, 3327, 3460, 3542, 4152, 4499, 4941, 5140, 5374
Criterion 2......	199, 316, 1073, 1330, 2517, 2918, 2986, 4385, 4592, 4759
6 2 −3 (17 members):	
Criterion 1......	5281
Criterion 2......	221, 339, 579, 742, 1105, 1234, 1353, 1364, 1533, 1737, 1984, 2309, 3194, 4041, 4058, 4958
3 −1 −1 (14 members):	
Criterion 1......	213, 485, 564, 578, 795, 947, 1427, 1642, 2042, 2465, 3534, 4106, 4426
Criterion 2......	560
8 −3 −3 (13 members):	
Criterion 1......	478, 639, 1654, 1844, 2263, 2413, 2573
Criterion 2......	450, 661, 1148, 3140, 3250, 4745
2 2 −1 (13 members):	
Criterion 1......	70, 258, 269, 839, 923, 1053, 2995, 3707, 4611
Criterion 2......	194, 2265, 3182, 3524
8 −4 −3 (13 members):	
Criterion 1......	10, 448, 468, 986, 3751, 4013, 5045, 5294
Criterion 2......	1487, 3507, 3866, 3922, 5204
6 1 −3 (12 members):	
Criterion 1......	936, 1125, 1247, 2016, 2293, 3683
Criterion 2......	152, 259, 928, 3847, 4412, 5082

TABLE 3

NUMBERED ASTEROIDS IN THE THREE-BODY MEAN MOTION RESONANCES: RESONANCES WITH 10 OR FEWER IDENTIFIED MEMBERS

Resonance	N_{mem}[a]	Members
7 −2 −3......	10	530, 835, 1109, 1371, 2127, 2582, 3025, 4208, 4741, 4814
6 −1 −2......	10	53, 1431, 1688,2292, 2660, 3426, 4349, 4602, 5147, 5199
3 −2 −1......	10	982, 2057, 2158, 2395, 2761, 3449, 3823, 4462, 5154, 5372
3 3 −2......	8	106, 555, 834, 1974, 2464, 3292, 4201, 4379
4 −4 −1......	8	22, 636, 1308, 2288, 2426, 2742, 4084, 5072
4 −1 −1......	7	443, 2175, 2440, 3370, 3739, 4246, 5313
5 5 −2......	7	728, 2112, 3807, 3824, 3986, 4563, 4984
2 3 −1......	6	1591, 2572, 2965, 3795, 4282, 5063
7 −4 −2......	6	166, 505, 789, 1132, 2474, 4175
9 −7 −3......	6	175, 381, 1838, 2414, 3278, 3859
7 −2 −2......	5	138, 1375, 1966, 3463, 3881
9 −5 −2......	5	2887, 3100, 3959, 3982, 4146
5 2 −2......	5	3742, 3861, 3904, 4612, 4639
4 −2 −1......	5	463, 3733, 4775, 5053, 5230
4 −3 −1......	4	792, 920, 1246, 3531
5 −1 −2......	4	494, 576, 952, 1319
8 −1 −3......	3	236, 417, 4182
4 6 −3......	3	882, 1519, 3230
13 −5 −4......	3	390, 771, 3662
5 −3 −3......	3	2563, 4098, 5337
3 1 −1......	2	1705, 3535
8 5 −3......	2	1083, 2422
7 −1 −2......	2	2733, 4817
7 −3 −2......	2	1430, 1658
5 1 −2......	2	64, 3874

[a] Number of identified asteroids.

TABLE 4

NUMBERED ASTEROIDS IN THE ORDINARY MEAN MOTION RESONANCES: RESONANCES WITH MORE THAN 10 IDENTIFIED MEMBERS

Resonance	Members
13:6 (17 members):	
Criterion 1......	86, 328, 784, 1331, 1520, 1958, 2352, 3234, 3420, 3499, 3504, 3878, 4176, 4301, 4609, 5043, 5070
7:2 (11 members):	
Criterion 1......	822, 2171, 3212, 3359, 3520, 3989, 4692, 4951, 5085, 5319
Criterion 2......	2460

TABLE 5

NUMBERED ASTEROIDS IN THE ORDINARY MEAN MOTION RESONANCES: RESONANCES WITH 10 OR FEWER IDENTIFIED MEMBERS

Resonance	N_{mem}[a]	Members
9:4	10	948, 1465, 1588, 1630, 3120, 3196, 4243, 4381, 4593, 5212
11:4	6	476, 1759, 1927, 2944, 3921, 3965
8:3	4	868, 2913, 3587, 4701
11:5	4	2630, 4049, 4439, 4915
16:7	4	747, 949, 2183, 3329
10:3	3	2762, 4336, 5236
7:3	2	677, 5324
12:5	2	1092, 1670

[a] Number of identified asteroids.

We have also performed an additional check on the possible influence of the perturbations of inner planets as concerns the identification of asteroids in resonances. From 10 asteroids previously identified in different three-body resonances in the integrations with four outer planets, the orbits of nine bodies had shown the same pattern of libration-circulation alternation of the resonant angle also in the integration with seven planets (without Mercury and Pluto), which was performed using the SWIFT integrator (Levison & Duncan 1994). In only one case was the asteroid previously classified as resonant with a librating resonant angle found to be a little outside the resonance, with the resonant angle slowly circulating.

In Figure 5, most of the chaotic asteroids appear to be concentrated between 3.1 and 3.25 AU, so it may be interesting to explore in detail the dynamics in this region.

Figure 6 shows the result of an extensive numerical simulation of many fictitious objects, still under the perturbations of the solely four outer planets. The initial angles and inclinations of the bodies were fixed at zero; the initial eccentricity was set to 0.05 (*bottom*), 0.15 (*middle*), and 0.25 (*top*). The initial semimajor axis of the particles ranged between 3.10 and 3.24 AU on a grid with 0.002 AU resolution. The integration span of 2 Myr allowed for the correct determination of the maximum Lyapunov exponent (*solid lines*) above the $10^{-5.3}$ yr^{-1} threshold. The dotted lines denote the relative change of the perihelion frequency over 1 Myr. The two indicators are complementary; the MLE tells us about the degree of the chaoticity, and the relative change of perihelion frequency should be related to the speed of chaotic diffusion. The scale of the y-axis is logarithmic.

As expected, one can see in Figure 6 that both ordinary and three-body resonances generate chaos and enhance the diffusion speed. We have identified all the resonances that are associated with an MLE of more than $10^{-4.5}$ yr^{-1}) in the middle panel. This required an additional short time integration and a check on the behavior of the corresponding resonant angles. The most important resonances at $e = 0.15$ are the $5\ -2\ -2$ and the $7\ -3\ -2$. Note that now, with the considerations of the three-body resonances, all the important generators of chaos seem to be identified. The different panels of Figure 6 also clearly show how the resonances shrink at low eccentricity. At $e = 0.05$, most of the investigated region is quite regular (MLE $< 10^{-5.3}$ yr^{-1}); on the other hand, at $e = 0.25$ the area is almost globally chaotic and is characterized by relatively fast diffusion.

Figure 6 also provides an additional estimate of how incomplete our identification was. From all the resonances seen at $e = 0.15$, only seven were considered in the identification process. Among the other resonances, there are about six with the Lyapunov time shorter than $10^{4.5}$ yr and many more with the Lyapunov time shorter than 10^5 yr.

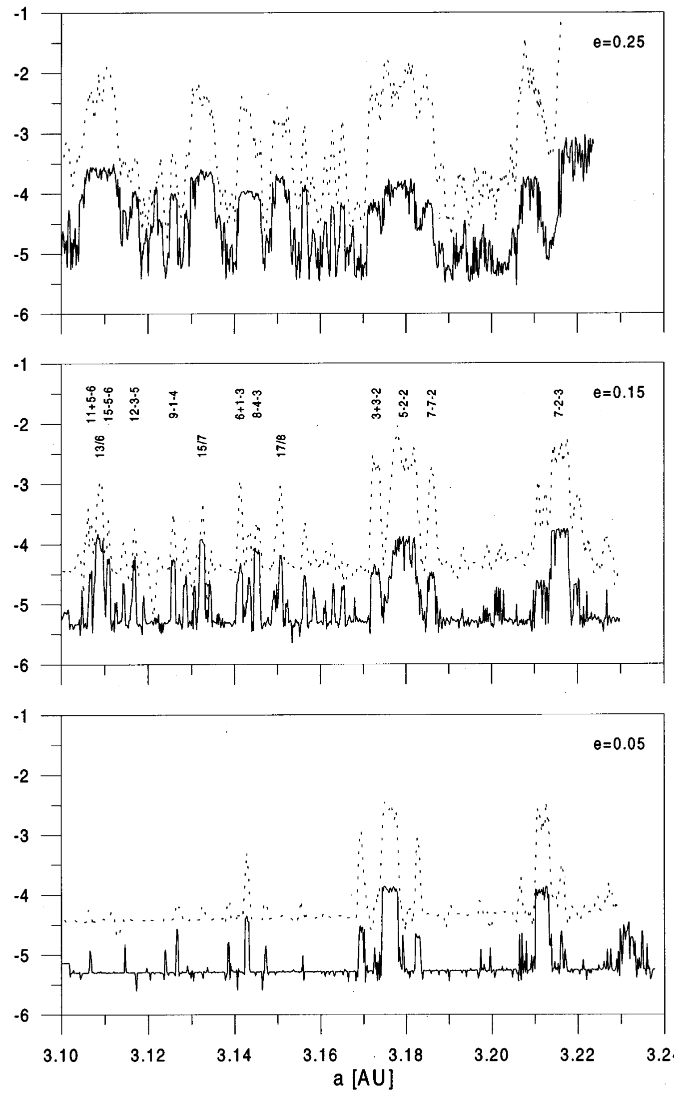

FIG. 6.—Same as Fig. 4, but for the region 3.1–3.23 AU at zero inclination and at three different initial eccentricities. In this region a large number of chaotic asteroids are located.

We have also performed a similar analysis for the entire asteroid belt. Between the 4:1 (~2.06 AU) and 2:1 (~3.27 AU) resonances with Jupiter, we have found about 100 mean motion resonances with the outer planets (either ordinary or three-body resonances) with Lyapunov times shorter than 10^5 yr, the latter computed for eccentricity equal to 0.1 and zero inclination. The total relative volume in phase space filled by these resonances increases with eccentricity, and in some parts of the asteroid belt, resonances may overlap and destroy all regular regions already at moderate eccentricity, as can be seen in Figure 6 for the interval 3.1 AU $< a <$ 3.22 AU and $e = 0.25$.

The inner part of the asteroid belt (2.2 AU $< a <$ 2.5 AU) appears to be much more regular than the part shown in Figure 6: the chaotic zones related to three-body or ordinary resonances are narrower and more separated, as one could expect looking at the different resonance density in Figure 3. We can therefore conclude that the concentration of the chaotic asteroids beyond 3.1 AU indicated by Figure

5 is a real feature and not an artifact introduced by the computation of the fast Lyapunov indicator. Hence, the majority of chaotic asteroids is to be expected in the region 3.1 AU $< a <$ 3.24 AU. However, this result changes if the perturbations of the inner planets are taken into account. In numerical simulations with seven planets, we have found that other mean motion resonances become important, the most numerous among them being the exterior resonances with Mars (e.g., 1:2, 4:7, 5:9, 7:13, 9:16).

5. CONCLUSION

It has been recently pointed out that some asteroids have a very short Lyapunov time despite not being related to any low-order secular resonance or mean motion resonance with Jupiter. 490 Veritas may be considered the chief representative of this class of asteroids of puzzling dynamical nature.

In this paper we have given numerical evidence that the behavior of these asteroids is due to the existence of a new class of resonances, neglected until now. These are the resonances among the mean motions of asteroid, Jupiter, and Saturn, which we call *three-body mean motion resonances*.

We have shown that, at equal strength, the three-body mean motion resonances are much more dense in the asteroid belt than the ordinary mean motion resonances with Jupiter. This is due to the fact that three frequencies instead of two are involved, so that the number of possible combinations is much larger.

The typical width of the three-body mean motion resonances of low order ranges from 10^{-4} to several 10^{-3} AU. The vast majority of the orbits in three-body resonances are chaotic, because each three-body resonance is in fact a multiplet of resonant components that strongly interact and overlap. The typical Lyapunov time associated with the low-order three-body resonances is of order 5×10^3 to 10^5 yr. Frequency analysis has allowed us to show that the secular frequencies have a relative change of 10^{-2} to 10^{-3} per million years, which indicates the possible existence of important chaotic diffusion in both eccentricity and inclination. In the case of 490 Veritas, this is confirmed by the long-term numerical integrations by Milani & Farinella (1994). The consideration of the three-body resonances, together with the ordinary mean motion and secular resonances, seems to be enough to understand the fine chaotic structure of the asteroid belt (see Fig. 6).

Finally, we have identified 255 numbered asteroids that are presently in some three-body mean motion resonance. Taking into account that numbered asteroids are about one-sixth of the presently known population, we estimate the total number of asteroids in three-body resonances to be ~ 1500. But of course, because of the incompleteness of the asteroid catalogs and the incompleteness of our method for the identification of resonant bodies, the real number should be much larger. This is a very important change in our understanding of the chaotic structure of the asteroid belt and of its generating mechanisms. Of course, the mean motion resonances with Jupiter with moderate order also play a relevant role in the generation of chaos, but we expect that they are of lesser importance for what concerns the real asteroid population, because the number of bodies associated with these resonances is smaller. The same is true for the outer mean motion resonances with Mars in the inner belt.

The fact that a large fraction of real main-belt asteroids is chaotic may have important implications for our understanding of the origin of Mars-crossing asteroids and near-Earth asteroids. Numerical simulations of the long-term dynamical behavior of resonant bodies are in progress and will be the subject of a forthcoming paper.

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